United Kingdom
Mathematics Trust

Intermediate Mathematical Challenge

Solutions 2023

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

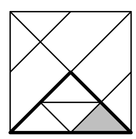
www.ukmt.org.uk

1. D Both 21 and 45 are multiples of 3. Also, both 16 and 28 are multiples of 4. However, 34 is neither a multiple of 3, nor a multiple of 4.

2. A By Pythagoras' Theorem, the length, in cm, of the third side of the triangle is $\sqrt{5^2 - 4^2} = 3$. So the area, in cm^2, of the triangle is $\frac{1}{2} \times 4 \times 3 = 6$.

3. B $1 - (2 - (3 - 4 - (5 - 6))) = 1 - (2 - (3 - 4 - (-1))) = 1 - (2 - (3 - 4 + 1)) = 1 - 2 = -1$.

4. D The diagram shows that a triangle whose area is one quarter of that of the square may be divided into four congruent triangles, one of which is the shaded triangle. So the required fraction is $\dfrac{1}{4} \times \dfrac{1}{4} = \dfrac{1}{16}$.



5. B It is given that $1^3 + 2^3 + 3^3 + 4^3 = 10^n$. So $10^n = 1 + 8 + 27 + 64 = 100 = 10 \times 10 = 10^2$. Hence $n = 2$.

6. E To draw a n by n square grid requires $(n + 1)$ horizontal lines and $(n + 1)$ vertical lines. So, in total, $2(n + 1)$ lines are required.

7. B 0.015% of 60 million $= \dfrac{0.015}{100} \times 60\,000\,000 = 0.015 \times 600\,000 = 15 \times 600 = 9000$.

8. E We are given that $\sqrt{\sqrt{x}} = 3$. Therefore $\sqrt{x} = 3^2 = 9$. So $x = 9^2 = 81$.

9. B Let the one further number be x. The table below shows possible values of x and the corresponding list of Merryn's numbers, written in ascending order.

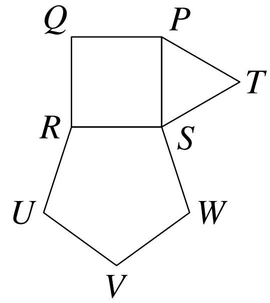
Range of x	Merryn's list
$x \leq 0$	x, 0, 2, 2, 3
$0 < x \leq 2$	0, x, 2, 2, 3
$2 < x \leq 3$	0, 2, 2, x, 3
$x > 3$	0, 2, 2, 3, x

In each case, the middle of Merryn's numbers is 2. So, whatever the value of the one further number, the median of the numbers is 2.

10. D There is only one two-digit power of 6, namely 36, so 1 down is 36. Also, there is only one four-digit power of 5, namely 3125. Therefore 1 across is 3125. There are two four-digit powers of 4, namely 1024 and 4096. However, we know from 1 down that the units digit of 2 across is 6, so 2 across is 4096. Hence the only two digits which do not appear in the completed crossnumber are 7 and 8. Their sum is 15.

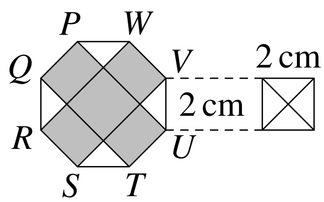
11. B After giving away one sixth of the jam and then one thirteenth of the remaining jam, Jill was left with twelve thirteenths of five sixths of the original weight of jam. Now $\frac{12}{13} \times \frac{5}{6} = \frac{60}{78} = \frac{10}{13}$. So ten thirteenths of the original weight of jam was 1 kg. Hence Jill had 1.3 kg of jam in the jar at the start.

12. E The four angles at point S include an interior angle of each of an equilateral triangle, a square and a regular pentagon. These are $60°$, $90°$ and $108°$ respectively. As the sum of the angles which meet at a point is $360°$, $\angle TSW$ is $(360 - (60 + 90 + 108))° = 102°$.
The equilateral triangle and the square have side PS in common and the square and the regular pentagon have side RS in common, so the triangle, square and pentagon have equal side-lengths. Therefore $SW = ST$ and hence triangle SWT is isosceles.
So $\angle WTS = \angle TWS = \frac{1}{2}(180 - 102)° = 39°$.



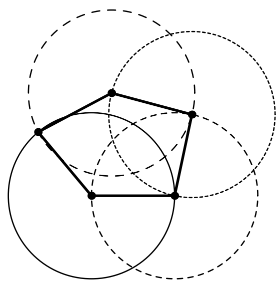
13. A The mean of p and q is 13. Therefore $p + q = 2 \times 13 = 26$. Similarly, $q + r = 2 \times 16 = 32$ and $r + p = 2 \times 7 = 14$. Adding these three equations gives $2p + 2q + 2r = 26 + 32 + 14 = 72$. Hence $p + q + r = 36$. So the mean of p, q and r is $\frac{p + q + r}{3} = \frac{36}{3} = 12$.

14. C The diagram on the left shows regular octagon $PQRSTUVW$ and the shaded rectangles, $PQTU$ and $RSVW$. The right-hand diagram shows how the four unshaded triangles fit together to form a square of side-length 2 cm. So the total area of these four triangles is $4\,\text{cm}^2$.



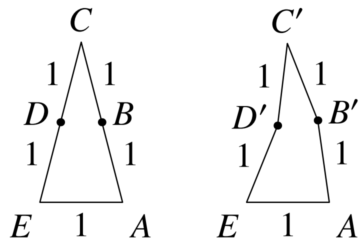
15. A A triangle with all sides the same length is an equilateral triangle, which has three equal interior angles. So the triangle described does not exist.
A quadrilateral with all sides the same length is a rhombus, which has two pairs of equal interior angles, or a square, which has four equal interior angles. So the quadrilateral described does not exist.
However, it is possible for a pentagon to have all its sides the same length and yet have five different interior angles.

To demonstrate this, the diagram shows four circles of equal radius. The centre of each circle lies on the circumference of one or more of the other circles. Four of the vertices of the pentagon shown are the centres of the circles and the fifth vertex is the point of intersection of the two circles on the left of the diagram. So the five sides of the pentagon are equal in length. By varying the positions of the circles relative to each other, the interior angles of the pentagon may be changed and the diagram shows one configuration in which these interior angles are different. It is left to the reader to draw such a configuration and to check that the interior angles of the pentagon are indeed all different.



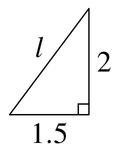
An alternative solution.

The left-hand diagram shows isosceles triangle ACE in which AE has length 1, while AC and EC both have length 2. The midpoints of AC and EC are B and D respectively. In the right-hand diagram, B, C and D have been moved very slightly to positions denoted by B', C' and D' such that AB', $B'C'$, $C'D'$ and $D'E$ are all of length 1. Hence $AB'C'D'E$ is a pentagon in which all of the sides have the same length. However, the interior angles at A and E are now different from each other as the former has increased and the latter decreased. Also, those at B' and D' are different from each other as the former is slightly smaller than 180° and the latter slightly greater. Any change in the interior angle at C is very slight, so, as can be seen, the interior angles of the pentagon are all different.



16. A Let the lengths, in cm, of the sides of the right-angled triangle be p, q and r, where r is the length of the hypotenuse. The area of the right-angled triangle is $\frac{1}{2}pq$.
We are told that $p + q + r = 16$ and $p^2 + q^2 + r^2 = 98$.
By Pythagoras' Theorem, $p^2 + q^2 = r^2$ so $2r^2 = 98$. Hence $r = 7$.
Therefore $p + q = 16 - 7 = 9$. So $(p + q)^2 - (p^2 + q^2) = 9^2 - 49 = 81 - 49 = 32$.
Hence $p^2 + 2pq + q^2 - p^2 - q^2 = 32$, so $pq = 16$.
Therefore the area of the right-angled triangle is $\frac{1}{2}pq = \frac{1}{2} \times 16 = 8$.

17. C First note that the perimeter of the rectangle is $2(2 + 3) = 10$.
The diagram shows one of the four congruent triangles into which the rectangle is divided.



By Pythagoras' Theorem, $l^2 = 2^2 + 1.5^2 = 4 + 2.25 = 6.25$.
Therefore $l = \sqrt{6.25} = 2.5$. So the perimeter of the rhombus is $4 \times 2.5 = 10$.
Hence the ratio of the two perimeters is $10 : 10 = 1 : 1$.

18. B Let n^2 be a square which is 4 greater than a prime, p.
Then $n^2 = p + 4$. So $p = n^2 - 4 = (n + 2)(n - 2)$.
However, as p is prime, its only factors are p and 1. So $n + 2 = p$ and $n - 2 = 1$.
Hence $n = 3$ and therefore $p = 3 + 2 = 5$.
So the only square which is 4 greater than a prime is $3^2 = 9$. It is 4 greater than 5, which is prime.

19. E Multiplying the numerator and denominator by $n + 3$:

$$\frac{n}{n + 1 - \dfrac{n+2}{n+3}} = \frac{n(n+3)}{(n+1)(n+3) - (n+2)} = \frac{n^2 + 3n}{n^2 + 4n + 3 - n - 2} = \frac{n^2 + 3n}{n^2 + 3n + 1}.$$

Therefore the difference between the numerator and denominator of the simplified fraction is 1.

20. B There are 36 different equally likely possible outcomes (a, b) for the two dice. The number of these which involve only 1, 2, 4, 5 or 6 is $5 \times 5 = 25$. So there are just eleven involving a 3, one of which is (3, 3). So the probability that both are 3, given that at least one is 3, is $\dfrac{1}{11}$.

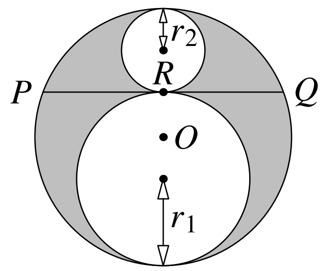
21. A As the semicircles intersect on the hypotenuse of the triangle, the shaded area, in square units, is the sum of the areas of the two semicircles minus the area of the right-angled triangle, that is $\frac{1}{2}\pi \times 3^2 + \frac{1}{2}\pi \times 4^2 - \frac{1}{2} \times 6 \times 8 = \frac{25\pi}{2} - 24$.

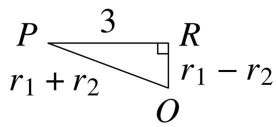
22. E Adding both equations gives: $1000x + 1000y = 5000$. So $x + y = 5$.
Subtracting the first equation from the second gives: $954x - 954y = 954$. Therefore $x - y = 1$.
Hence $x^2 - y^2 = (x + y)(x - y) = 5 \times 1 = 5$.

23. D The product of the positive integers from 1 to 9 inclusive is $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$.
As the product of prime factors, this product is $2 \times 3 \times 2^2 \times 5 \times 2 \times 3 \times 7 \times 2^3 \times 3^2 = 2^7 \times 3^4 \times 5 \times 7$.
Therefore, for the product of six of the integers to be a square, it is necessary to discard 5, 7 and an odd power of 2 smaller than 10, that is 2 or 8.
Let $p < q$. Then, discarding 8 gives $p^2 = 2^4 \times 3^4$. So $p = 2^2 \times 3^2 = 4 \times 9 = 36$.
Similarly, discarding 2 gives $q^2 = 2^6 \times 3^4$. Therefore $q = 2^3 \times 3^2 = 8 \times 9 = 72$.
Hence $p + q = 36 + 72 = 108$.

24. C The gradient of VQ is $-\dfrac{PV}{QP} = -\dfrac{d}{b}$. The gradient of XQ is $-\dfrac{TX}{QT} = -\dfrac{a}{b - c}$.
Therefore, for Q, X and V to lie on a straight line, it is necessary that $\dfrac{d}{b} = \dfrac{a}{b - c}$.
Hence $ab = bd - cd$, which simplifies to $\dfrac{ab + cd}{bd} = 1$. This in turn simplifies to $\dfrac{a}{d} + \dfrac{c}{b} = 1$.

25. D Let the centre of the large circle and the point where the two smaller circles touch be O and R respectively. Let the radii of the two smaller circles be r_1 and r_2, where $r_1 \geq r_2$. Then the radius of the large circle is $r_1 + r_2$. Consider triangle OPR: $PR = \frac{1}{2}PQ = 3$; $OP = r_1 + r_2$ and $OR = r_1 + r_2 - 2r_2 = r_1 - r_2$.





Therefore, by Pythagoras' Theorem, $(r_1 + r_2)^2 = (r_1 - r_2)^2 + 3^2$.
So $r_1^2 + 2r_1r_2 + r_2^2 = r_1^2 - 2r_1r_2 + r_2^2 + 9$. Hence $r_1r_2 = \dfrac{9}{4}$.
Now the area of the shaded region is $\pi(r_1 + r_2)^2 - \pi r_1^2 - \pi r_2^2 = \pi(2r_1r_2) = \pi \times 2 \times \dfrac{9}{4} = \dfrac{9\pi}{2}$.

(*Note that the listed possible solutions show that the answer is a value independent of how the diagram is configured. So a simpler method is to consider the case when PQ is a diameter of the large circle and the two small circles each have diameters of length 3. It is left to the reader to check that this configuration gives the same answer as does the general solution above.*)